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BOUNDING THE VERTEX COVER NUMBER OF A HYPERGRAPH

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For a hypergraph H, we denote by
(i) $\tau(H)$ the minimum k such that some set of k vertices meets all the edges,
(ii) $\nu(H)$ the maximum k such that some k edges are pairwise disjoint, and
(iii) $\lambda(H)$ the maximum $k \geq 2$ such that the incidence matrix of H has as a submatrix the transpose of the incidence matrix of the complete graph K_k.
We show that $\tau(H)$ is bounded above by a function of $\nu(H)$ and $\lambda(H)$, and indeed that if $\lambda(H)$ is bounded by a constant then $\tau(H)$ is at most a polynomial function of $\nu(H)$.

1. Introduction

A *hypergraph* H consists of a finite set $V(H)$ of *vertices* and a set $E(H)$ of subsets of $V(H)$ called *edges*. If every edge is non-empty, we denote by $\nu(H)$ the maximum k such that there are edges $e_1,\dots,e_k$ which are pairwise disjoint, and by $\tau(H)$ the minimum k such that there is a set $X \subseteq V(H)$ with $|X| = k$ which intersects every edge.

It is immediate that $\tau(H) \geq \nu(H)$, and in fact the "fractional" versions of these two parameters are equal by linear programming duality. The non-equality of $\tau(H)$ and $\nu(H)$ in general is a fundamental property (and critical difficulty) of *integer* programming, and more particularly of linear optimization in which the variables are constrained to take only the values 0 or 1.

In fact there are several interesting types of hypergraph for which equality holds; but typically $\tau(H) > \nu(H)$ and indeed $\tau(H)$ is not even bounded by any function of $\nu(H)$. To see this, let K_n be the complete graph on n vertices, and let H be the hypergraph whose vertices are the edges of K_n and whose edges are the stars of K_n (a *star* is the set of all edges incident with any one vertex). Then $\nu(H)=1$ and $\tau(H) \geq n/2$.

One way to state our main result is to say that any such example (that is, any sequence of hypergraphs with ν bounded and τ unbounded) contains this one, the hypergraph of stars of K_n, for arbitrarily large n. Let us formulate this more precisely. We define $\lambda(H)$ (or just λ, when only one hypergraph is under consideration) to be the maximum $k \geq 2$ such that there are edges $e_1,\dots,e_k$ with

AMS subject classification codes (1991): 05 C 65, 05 C 35 the property that for $1 \le i < j \le k$, some vertex v satisfies

$$\{h:\ 1 \le h \le k,\ v \in e_h\} = \{i, j\}.$$

We put $\lambda(H) = 2$ if there is no such k. Our main result is

(1.1) *For any hypergraph with all edges non-empty,*

$$\tau \le 11\lambda^2(\lambda + \nu + 3)\binom{\lambda + \nu}{\nu}^2.$$

Here are two applications of (1.1).

Application 1. Let V be a finite set of points in the plane, and let $\mathbf{C}$ be a finite set of rectangles, each with sides parallel to the x- and y-axes, and each containing a member of V. Let k be a positive integer. Then either

(i) there are $k+1$ members of $\mathbf{C}$ such that no member of V is in more than one of them, or

(ii) there exists $Y \subseteq V$ with $|Y| \le (k+63)^{63}$ meeting every member of $\mathbf{C}$.

For, let H be the hypergraph with $V(H) = V$ and $E(H) = \{C \cap V\ :\ C \in \mathbf{C}\}$. Suppose that $\lambda(H) \ge 64$. Then there are 64 members $C_1, \dots, C_{64}$ of H such that for $1 \le i < j \le 64$ there exists $v_{ij} \in V$ belonging to C_i and to C_j and not to any of the other 62 rectangles. In particular $C_i \cap C_j \ne \emptyset$ $(1 \le i < j \le 64)$ and so, by an easily established property of aligned rectangles, $\bigcap_{1 \le i < j \le 64} C_i \ne \emptyset$. We may assume, therefore, that each C_i contains the origin (0,0).

For $1 \le i < j \le 64$, let us say that the pair ij has color 1, 2, 3 or 4 depending on which quadrant contains v_{ij} (choosing arbitrarily if v_{ij} belongs to more than one quadrant). We obtain a partition of $E(K_{64})$ into four sets, and so by [3], there exist $i_1 < i_2 < i_3$ such that $i_1 i_2$, $i_2 i_3$ and $i_1 i_3$ all have the same colors. But it is easy to see that this is impossible. Hence $\lambda(H) \le 63$. By (1.1),

$$\tau \le 11 \cdot 63^2(\nu + 66)\binom{\nu + 63}{63}^2 \le (\nu + 63)^{127}.$$

This bound has now been improved to $2 \cdot 10^5 \nu^8$ by Pach and Törőcsik [6].

Application 2. Let G be a graph drawn in a connected compact surface Σ, let $X \subseteq V(G)$, and let $k \ge 0$ be an integer. Let n be maximum so that K_n can be drawn in Σ. Then either there are $k+1$ members of X, no two on a common region, or there are $\le (n+k)^{2n+1}$ regions of G, with X in the closure of their union.

For let H be the hypergraph with $V(H)$ the set of all regions of G and with edges e_v $(v \in X)$, where e_v denotes the set of regions incident with v. Now $\lambda(H) \le n$; for there exist λ members of X, such that for any two of them some region is incident with those two and no others, and so K_λ can be drawn in Σ. From (1.1),

$$\tau(H) \le 11n^2(n + \nu + 3)\binom{n + \nu}{n}^2 \le (n + \nu)^{2n+1},$$

since $n \ge 4$.

Actually, this is a weak form of a result of Bienstock and Dean [1].

For $d \geq 1$, let us define $\nu_d(H)$ to be the maximum k such that there are edges $e_1, \ldots, e_k$ with no vertex in more than d of them. (Thus $\nu_1(H) = \nu(H)$.) Again, for any given d, τ is not bounded by any function of ν_d, and indeed ν_d+1 is not bounded by a function of ν_d (the latter is a stronger statement since $\tau(H) \geq \nu_{d+1}/(d+1)$). For example, let K be the hypergraph of all $(d+1)$-subsets of an n-set where $n \geq d+1$, and let H be the hypergraph with $V(H) = E(K)$ and

$$E(H) = \{\{e \ : \ v \in e \in E(K)\} \ : \ v \in V(K)\}.$$

Then $\nu_d(H) = d$ and $\nu_{d+1}(H) = n$.

We can extend (1.1) to a result for $\nu_d(H)$ as follows. For $d \geq 1$, we define $\lambda_d(H)$ to be the maximum $k \geq d$ such that there are edges $e_1, \ldots, e_k$ with the property that for all $X \subseteq \{1, \ldots, k\}$ with $|X| = d$, some vertex v satisfies

$$\{h \ : \ 1 \leq h \leq k, \quad v \in e_h\} = X.$$

(If there is no such k we define $\lambda_d(H) = d$.) Thus, $\lambda_2(H) = \lambda(H)$. We shall show that for any $d \geq 2$ and any hypergraph H, $\tau(H)$ (and thus also $\nu_d(H)$) is bounded above by a function of $\lambda_d(H)$ and $\nu_{d-1}(H)$; and then (1.1) will be obtained by setting $d = 2$.

2. Faithful sets

Let H be a hypergraph and $d \geq 0$ an integer. We say that $Y \subseteq V(H)$ is *d-faithful* if for every $X \subseteq Y$ with $|X| = d$ which is a subset of an edge of H, there is an edge $e \in E(H)$ with $e \cap Y = X$. To prove (1.1) we need some results about d-faithful sets. We begin with the following.

(2.1) *Let $\varepsilon \geq 0$, let H be a hypergraph so that $|e| \leq \varepsilon |V(H)|$ for every edge e, let $d \geq 0$ be an integer, and let k be an integer with $d+1 \leq k \leq |V(H)|$. Then there is a d-faithful set of cardinality $\geq k\left(1 - \varepsilon\binom{k-1}{d}\right)$.*

Proof. We may assume that $\varepsilon \leq 1$, for otherwise the result is trivial, since the empty set is d-faithful. Let

$$\mathsf{A} = \{X \subseteq V(H) \ : \ |X| = d \text{ and } X \subseteq e \text{ for some edge } e\}.$$

For each $X \in \mathsf{A}$, let $f(X) \subseteq V(H) - X$ be such that $X \cup f(X) \in E(H)$. Thus, each $f(X)$ has cardinality $\leq \varepsilon n - d \leq \varepsilon(n-d)$, where $n = |V(H)|$. For each $Z \subseteq V(H)$, let

$$\beta(Z) = \{z \in Z \ : \ \text{there exists } X \in \mathsf{A} \text{ with } X \subseteq Z \text{ and } z \in f(X)\}.$$

Now for each $X \in \mathsf{A}$ and each $z \in f(X)$ there are $\binom{n-d-1}{k-d-1}$ k-sets in $V(H)$ which include $X \cup \{z\}$, since $d+1 \leq k \leq |V(H)|$. Thus, summing over all $Z \subseteq V(H)$ with $|Z| = k$, we obtain

$$\sum_Z |\beta(Z)| \leq \sum_Z \sum (|f(X) \cap Z| \ : \ X \in \mathsf{A}, \ X \subseteq Z) = \sum_{X \in \mathsf{A}} \sum_{z \in f(X)} \binom{n-d-1}{k-d-1}$$
$$\leq \sum_{X \in \mathsf{A}} \varepsilon(n-d)\binom{n-d-1}{k-d-1} \leq \binom{n}{d}\varepsilon(n-d)\binom{n-d-1}{k-d-1} = \varepsilon k \binom{k-1}{d}\binom{n}{k}.$$

Consequently there exists $Z \subseteq V(H)$ with $|Z| = k$ such that $|\beta(Z)| \le \varepsilon k\binom{k-1}{d}$. Let $Y = Z - \beta(Z)$. Then Y is d-faithful, for if $X \subseteq Y$ and $X \in \mathsf{A}$ then $f(X) \subseteq \beta(Z)$ and so $(X \cup f(X)) \cap Y = X$. But

$$|Y| = |Z| - |\beta(Z)| \ge k - \varepsilon k\binom{k-1}{d}$$

as required. ∎

We deduce

(2.2) *Let $\varepsilon \ge 0$, let H be a hypergraph such that $1 \le \varepsilon|V(H)|$ and $|e| \le \varepsilon|V(H)|$ for every edge e, and let d be an integer with $1 \le d \le |V(H)|$. Then there is a d-faithful set of cardinality*

$$> \frac{d}{d+1}\left(\frac{d!}{d+1}\varepsilon^{-1}\right)^{\frac{1}{d}}.$$

Proof. Let k be the smallest integer with $k > \left(\frac{d!}{d+1}\varepsilon^{-1}\right)^{\frac{1}{d}}$, and let $n = |V(H)|$. Since $d \le n$, there is a subset of $V(H)$ of cardinality d, which is trivially d-faithful. We may assume that it does not satisfy the theorem, and consequently that $d+1 \le k$. Now

$$\frac{d!}{d+1}n^{1-d} < d^{d-1}n^{1-d} \le 1 \le \varepsilon n,$$

and so

$$n > \left(\frac{d!}{d+1}\varepsilon^{-1}\right)^{1/d};$$

and hence $k \le n$, from the definition of k. We have shown then that $d+1 \le k \le n$. From (2.1) there is a d-faithful set X of cardinality at least $k\left(1 - \varepsilon\binom{k-1}{d}\right)$. But by the minimality of k,

$$k - 1 \le \left(\frac{d!}{d+1}\varepsilon^{-1}\right)^{1/d},$$

and so

$$\binom{k-1}{d} \le \frac{(k-1)^d}{d!} \le \frac{\varepsilon^{-1}}{d+1};$$

and consequently

$$|X| \ge k\left(1 - \varepsilon\binom{k-1}{d}\right) \ge k\left(1 - \frac{1}{d+1}\right) = \frac{d}{d+1}k.$$

The result follows. ∎

3. Ramsey's theorem

Let k, r, $s \geq 0$ be integers. We denote by $R(k,r,s)$ the smallest integer $n \geq 0$ such that, if V is a set of cardinality n, then for every set A of k-subsets of V, either

(i) there exists $X \subseteq V$ with $|X| > r$ such that every k-subset of X belongs to A, or

(ii) there exists $Y \subseteq V$ with $|Y| > s$ such that no k-subset of Y belongs to A.

Ramsey's theorem asserts that $R(k,r,s)$ exists, for all k, r, s. Trivially $R(1,r,s) = r+s+1$, and it is easy to see that $R(2,r,s) \leq \binom{r+s}{r}$.

Let H be a hypergraph, and $d \geq 1$ an integer. We denote by α_d or $\alpha_d(H)$ the maximum of $\sum_{v \in V(H)} w(v)$, taken over all functions $w : V(H) \to \mathbb{Z}_+$ (the non-negative integers) such that $\sum_{v \in e} w(v) \leq d$ for every $e \in E(H)$. Thus, α_d is defined if and only if every vertex belongs to an edge.

A subset $Z \subseteq V(H)$ is said to be *d-complete* (in H), where $d \geq 0$ is an integer, if $|Z| \geq d$ and for every $X \subseteq Z$ with $|X| = d$ there is an edge $e \in E(H)$ with $e \cap Z = X$. We define ϕ_d or $\phi_d(H)$ to be the maximum cardinality of a d-complete subset of $V(H)$, or $\phi_d = d$ if there is no such subset.

(3.1) *Let H be a hypergraph with $V(H) \neq \emptyset$ such that every vertex belongs to an edge, and let $d \geq 2$ be an integer. Then there exists $e \in E(H)$ such that*

$$\frac{|e|}{|V(H)|} > \frac{d!d^d}{(d+1)^{d+1}} R(d, \phi_d, \alpha_{d-1})^{-d}.$$

Proof. Let $|V(H)| = n$, and let

$$\varepsilon = \frac{d!d^d}{(d+1)^{d+1}} R(d, \phi_d, \alpha_{d-1})^{-d}.$$

Now $\phi_d \geq d \geq d-1$, and $\alpha_{d-1} \geq d-1$ since $V(H) \neq \emptyset$, and so $R(d,\phi_d,\alpha_{d-1}) \geq d$. Consequently, $\varepsilon \leq d^{-1}$. Suppose for a contradiction that $|e| \leq \varepsilon n$ for every edge e. Then $\varepsilon n \geq 1$ since H has a non-empty edge, and so $d \leq n$. By (2.2) there is a d-faithful set $Z \subseteq V(H)$ with

$$|Z| \geq \frac{d}{d+1} \left(\frac{d!}{d+1} \varepsilon^{-1} \right)^{\frac{1}{d}} = R(d, \phi_d, \alpha_{d-1}).$$

Let A be the set of all $X \subseteq Z$ with $|X| = d$ such that $X \subseteq e$ for some edge e. By definition of α_{d-1}, there does not exist $Y \subseteq Z$ with $|Y| > \alpha_{d-1}$ such that $|Y \cap e| \leq d-1$ for every edge $e \in E(H)$, and in particular there does not exist $Y \subseteq Z$ with $|Y| > \alpha_{d-1}$ including no member of A. By definition of $R(d,\phi_d,\alpha_{d-1})$, there exists $Y \subseteq Z$ with $|Y| > \phi_d$ such that A contains every d-subset of Y. We claim that Y is d-complete. For let $X \subseteq Y$ with $|X| = d$. Since $X \in \mathsf{A}$, there exists $e \in E(H)$ with $X \subseteq e$. Since $|X| = d$ and Z is d-faithful, e may be chosen so that $e \cap Z = X$, and in particular $e \cap Y = X$. Thus Y is d-complete. Yet $|Y| > \phi_d$, a contradiction. It follows that $|e| > \varepsilon n$ for some edge e. ∎

4. Multiplying vertices

Let H be a hypergraph and let $w : V(H) \to \mathbb{Z}_+$ be a function. We denote by H^w the hypergraph obtained as follows. For each $v \in V(H)$ let $W(v)$ be a set of cardinality $w(v)$, so that the sets $W(v)$ $(v \in V(H))$ are mutually disjoint. Let $V(H^w) = \bigcup_{v \in V(H)} W(v)$. Let the edges of H^w be the sets $\bigcup_{v \in e} W(v)$, as e ranges over $E(H)$. Thus, roughly, H^w is obtained from H by replacing each vertex v by $w(v)$ copies of itself.

(4.1) *Let H be a hypergraph and $w : V(H) \to \mathbb{Z}_+$ a function; and let $d \geq 1$ be an integer. Then $\phi_d(H^w) \leq \phi_d(H)$.*

Proof. Let the sets $W(v)$ $(v \in V(H))$ be as above. By definition, $\phi_d(H) \geq d$, and so we may assume that $\phi_d(H^w) \geq d+1$. It follows that there exists $Z \subseteq V(H^w)$ with $|Z| = \phi_d(H) \geq d+1$, such that Z is d-complete in H^w. We claim that

(i) $|Z \cap W(v)| \leq 1$ for each $v \in V(H)$.

For let $v \in V(H)$, and suppose that $|Z \cap W(v)| \geq 2$. Since $|Z| \geq d+1$ and $d \geq 1$, there exists $X \subseteq Z$ with $|X| = d$ such that $X \cap W(v) \neq \emptyset$ and $W(v) \cap Z \not\subseteq X$. Since Z is d-complete in H^w, there exists $e \in H^w$ with $e \cap Z = X$. Then $W(v) \subseteq e$, since $X \cap W(v) \neq \emptyset$; and so $W(v) \cap Z \subseteq e \cap Z = X$, a contradiction. This proves (i).

Let $Y = \{v \in V(H) : Z \cap W(v) \neq \emptyset\}$. By (i), $|Y| = |Z|$, and we claim that Y is d-complete in H. For let $X \subseteq Y$ with $|X| = d$, and let $X' = \bigcup \{Z \cap W(v) : v \in X\}$. Then $|X'| = |X| = d$ and $X' \subseteq Z$, and so since Z is d-complete in H, there exists $f \in E(H^w)$ with $f \cap Z = X'$. Let $e \in E(H)$ be such that $f = \bigcup \{W(v) : v \in e\}$. Then

$$\bigcup_{v \in e} (W(v) \cap Z) = X' = \bigcup_{v \in X} (W(v) \cap Z)$$

and since the sets $W(v) \cap Z$ $(v \in Y)$ are disjoint and non-empty, it follows that $e \cap Y = X \cap Y = X$. Hence Y is d-complete in H, and so

$$\phi_d(H^w) = |Z| = |Y| \leq \phi_d(H)$$

as required. ∎

Let H be a hypergraph. We denote by $\alpha^*(H)$ the maximum of $\sum_{v \in V(H)} w(v)$, taken over all functions $w : V(H) \to \mathbb{R}_+$ (the non-negative reals) such that $\sum_{v \in e} w(v) \leq 1$ for every $e \in E(H)$. Thus, $\alpha^*(H)$ is defined if and only if every vertex belongs to some edge. It is easy to see that

$$\lim_{d \to \infty} \frac{1}{d} \alpha_d(H) = \max_{d \geq 1} \frac{1}{d} \alpha_d(H) = \alpha^*(H).$$

From (3.1) and (4.1) we deduce

(4.2) *Let H be a hypergraph such that every vertex belongs to an edge, and let $d \geq 2$ be an integer. Then*

$$\alpha^*(H) \leq \frac{(d+1)^{d+1}}{d!d^d} R(d, \phi_d, \alpha_{d-1})^d.$$

Proof. There exists $w : V(H) \to \mathbb{R}_+$ such that $\sum_{v \in e} w(v) \leq 1$ for every edge e, and $\sum_{v \in V(H)} w(v) = \alpha^*(H)$; and w may be chosen rational-valued, as is easily seen. Let $N > 0$ be an integer such that $w'(v) = Nw(v)$ is an integer for all $v \in V(H)$. Define $H^{w'}$ using sets $W(v)$ $(v \in V(H))$. Now

$$|V(H^{w'})| = \sum_{v \in V(H)} w'(v) = \sum_{v \in V(H)} Nw(v) = N\alpha^*(H).$$

For each edge $e' \in E(H^{w'})$, let $e \in E(H)$ be such that $e' = \bigcup_{v \in e} W(v)$; then

$$|e'| = \sum_{v \in e} w'(v) = \sum_{v \in e} Nw(v) \leq N.$$

Consequently,

$$\frac{|e'|}{|V(H^{w'})|} \leq \alpha^*(H)^{-1}$$

for every $e' \in E(H^{w'})$ (for we may assume that $\alpha^*(H) > 0$, since otherwise the result is trivial).

By (3.1) applied to $H^{w'}$,

$$\alpha^*(H)^{-1} > \frac{d!d^d}{(d+1)^{d+1}} R(d, \phi_d(H^{w'}), \alpha_{d-1}(H^{w'}))^{-d}.$$

But $\phi_d(H^{w'}) \leq \phi_d(H)$ by (4.1), and clearly $\alpha_{d-1}(H^{w'}) \leq \alpha_{d-1}(H)$, and so

$$R(d, \phi_d(H), \alpha_{d-1}(H)) \geq R(d, \phi_d(H^{w'}), \alpha_{d-1}(H^{w'})).$$

It follows that

$$\alpha^*(H) < \frac{(d+1)^{d+1}}{d!d^d} R(d, \phi_d(H), \alpha_{d-1}(H))^d$$

as required. ∎

5. Vapnik–Chervonenkis dimension

For a hypergraph H with $|E(H)| \geq 1$, we denote by δ or $\delta(H)$ the *Vapnik–Chervonenkis dimension* of H, that is, the maximum cardinality of a subset $Z \subseteq V(H)$ such that for every $X \subseteq Z$ there exists $e \in E(H)$ with $e \cap Z = X$. (See [2,4,8].) We see that $\delta(H) \geq 0$, and $\delta(H) = 0$ if and only if $|E(H)| = 1$. We need a strengthening of a result of Haussler and Welzl [4], due to Blumer, Ehrenfeucht, Haussler and Warmuth [2]. (See also [5] for a more general result with the strongest known bounds.) For the reader's convenience (and since what we really want is a slight variant of the latter result) we give a proof in full. (Logarithms are to base 2.)

(5.1) *Let H be a hypergraph with $|E(H)| \geq 2$, and let $0 < \varepsilon \leq 1$, such that $|e| \geq \varepsilon|V(H)|$ for every edge $e \in E(H)$. Then $\tau \leq 2\delta\varepsilon^{-1}\log(11\varepsilon^{-1})$.*

We use the following lemma of [8].

(5.2) *Let H be a hypergraph with $|V(H)| = n$ and $E(H) \neq \emptyset$. Then*

$$|E(H)| \leq \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{\delta}.$$

Proof. We proceed by induction on n. If $n \leq \delta$ the result is clear, and so we assume that $n > \delta$ and in particular $n > 0$. Choose $v \in V(H)$. Let H_1 be the hypergraph with $V(H_1) = V(H) - \{v\}$ and

$$E(H_1) = \{e \cap (V(H) - \{v\}) \ : \ e \in E(H)\};$$

then $\delta(H_1) \leq \delta(H)$, and so $|E(H_1)| \leq \binom{n-1}{0} + \ldots + \binom{n-1}{\delta}$ from the inductive hypothesis. Let H_2 be the hypergraph with $V(H_2) = V(H) - \{v\}$ and

$$E(H_2) = \{e \in E(H) \ : \ v \notin e \text{ and } e \cup \{v\} \in E(H)\}.$$

Then $|E(H_1)| + |E(H_2)| = |E(H)|$. If $E(H_2) = \emptyset$ then $|E(H_1)| = |E(H)|$ and the result follows, and so we assume $E(H_2) \neq \emptyset$. Hence $\delta(H_2)$ exists and $\delta(H_2) \leq \delta(H) - 1$. Consequently, $|E(H_2)| \leq \binom{n-1}{0} + \ldots + \binom{n-1}{\delta-1}$, and the result follows by adding. ∎

Proof of (5.1). Let $V(H) = V$ and $|V| = n$. Let $\tau = 2\delta\varepsilon^{-1}\log(c\varepsilon^{-1})$; we shall prove that $c < 11$, from which the theorem follows. Let $t = \tau - 1$. We may assume that $t \geq \delta$, for otherwise the theorem holds.

For each $e \in E(H)$, let A_e be the set of all $(x_1, \ldots, x_{2t}) \in V^{2t}$ such that $x_1, \ldots, x_t \notin e$ and $|\{x_{t+1}, \ldots, x_{2t}\} \cap e| \geq \varepsilon t - 1$; and let $\mathsf{A} = \bigcup(\mathsf{A}_e \ : \ e \in E(H))$. Let $(x_1, \ldots, x_t) \in V^t$. Since $t < \tau$, it follows that there exists $e \in E(H)$ with $\{x_1, \ldots, x_t\} \cap e = \emptyset$.

We now make use of the fact that the median of a binomial distribution is within one of its mean; this can be deduced, for example, from inequality (v) on the bottom of page 404 of [7]. Since $|e| \geq \varepsilon t$, it follows that there are at least $n^t/2$

sequences $(x_{t+1},\dots,x_{2t})$ such that $(x_1,\dots,x_{2t})\in \mathsf{A}_e$. Summing over all $(x_1,\dots,x_t)$, we deduce that $|\mathsf{A}|\geq n^{2t}/2$.

For $(x_1,\dots,x_{2t})\in V^{2t}$, its support is the function μ with domain V, where for $v\in V$, $\mu(v)$ is the number of values of i $(1\leq i\leq 2t)$ with $x_i=v$. Let μ be the support of a member of V^{2t}. Let S be the set of all $(x_1,\dots,x_{2t})\in V^{2t}$ with support μ, and let $\mathsf{S}_e=\mathsf{S}\cap \mathsf{A}_e$ for $e\in E(H)$. We claim that for each $e\in E(H)$, $|\mathsf{S}_e|\leq 2^{1-\varepsilon t}|\mathsf{S}|$. For let $k=\sum_{v\in e}\mu(v)$. If $k>t$ or $k<\varepsilon t-1$ then $\mathsf{S}_e=\emptyset$ and the inequality holds; and otherwise

$$|\mathsf{S}_e|=\frac{t(t-1)\dots(t-k+1)}{2t(2t-1)\dots(2t-k+1)}|\mathsf{S}|\leq 2^{-k}|\mathsf{S}|\leq 2^{1-\varepsilon t}|\mathsf{S}|.$$

This proves the claim. Now let $X=\{v\in V\ :\ \mu(v)>0\}$. If e, $f\in E(H)$ and $e\cap X=f\cap X$ then $\mathsf{S}_e=\mathsf{S}_f$; and by (5.2), since $|X|\leq 2t$ and $t\geq\delta\geq 1$, there are at most

$$\binom{2t}{0}+\dots+\binom{2t}{\delta}\leq\frac{(2t+2)^\delta}{(\delta-1)!}\leq\frac{1}{4}(8\tau\delta^{-1})^\delta$$

edges $e\in E(H)$ with the sets $e\cap X$ distinct. Hence there are at most that many distinct sets S_e, and so

$$\left|\bigcup_{e\in E(H)}\mathsf{S}_e\right|\leq\frac{1}{4}(8\tau\delta^{-1})^\delta 2^{1-\varepsilon t}|\mathsf{S}|.$$

Every member of A belongs to $\bigcup_{e\in E(H)}\mathsf{S}_e$ for some choice of μ, and so, summing over all μ, we deduce that

$$|\mathsf{A}|\leq\frac{1}{4}(8\tau\delta^{-1})^\delta 2^{1-\varepsilon t}n^{2t}.$$

Since $|\mathsf{A}|\geq n^{2t}/2$, it follows that $(8\tau\delta^{-1})^\delta 2^{-\varepsilon t}\geq 1$, and hence

$$8\tau\delta^{-1}\geq 2^{\varepsilon t\delta^{-1}}\geq 2^{\varepsilon\tau\delta^{-1}}/2.$$

Now $\tau=2\delta\varepsilon^{-1}\log(c\varepsilon^{-1})$, and so

$$16\varepsilon^{-1}\log(c\varepsilon^{-1})\geq c^2\varepsilon^{-2}/2.$$

We wish to prove that $c<11$, and we may therefore assume that $c\varepsilon^{-1}>c>e$, and hence that

$$\frac{\log(c\varepsilon^{-1})}{c\varepsilon^{-1}}<\frac{\log(c)}{c}.$$

Consequently, $32\log(c)\geq c^2$, and hence $c<11$. The result follows. ∎

We also need the following. (H^w is defined in Section 4.)

(5.3) *Let H be a hypergraph with $E(H) \neq \emptyset$, and let $w : V(H) \to \mathbb{Z}_+$. Then $\delta(H^w) \leq \delta(H)$.*

Proof. Let the sets $W(v)$ $(v \in V(H))$ be as in the start of Section 4. Now $E(H^w) \neq \emptyset$; let $Z \subseteq V(H^w)$ with $|Z| = \delta(H^w)$ such that for all $X \subseteq Z$ there exists $f \in E(H^w)$ with $f \cap Z = X$. As in (4.1), $|Z \cap W(v)| \leq 1$ for all $v \in V(H)$. Let

$$Y = \{v \in V(H) : |Z \cap W(v)| = 1\};$$

then the result follows as in (4.1). ∎

The following method of reformulating (5.1) is due independently to L. Lovász (private communication) and to Komlós et al. [5]. If H is a hypergraph with all edges non-empty, we define τ^* or $\tau^*(H)$ to be the minimum of $\sum_{v \in V(H)} w(v)$, taken over all functions $w : V(H) \to \mathbb{R}_+$ such that $\sum_{v \in e} w(v) \geq 1$ for every $e \in E(H)$.

(5.4) *Let H be a hypergraph with $|E(H)| \geq 2$ and with every edge non-empty. Then*

$$\tau \leq 2\delta\tau^* \log(11\tau^*).$$

Proof. Choose $w^* : V(H) \to \mathbb{R}_+$ such that $\sum_{v \in V(H)} w^*(v) = \tau^*(H)$ and $\sum_{v \in e} w^*(v) \geq 1$ for every $e \in E(H)$. We may choose w^* rational-valued. Let $N > 0$ be an integer so that $w(v) = Nw^*(v)$ is an integer for all $v \in V(H)$. Then

$$|V(H^w)| = \sum_{v \in V(H)} w(v) = N \sum_{v \in V(H)} w^*(v) = N\tau^*(H)$$

and for every edge $f = \bigcup(W(v) : v \in e)$ of H^w (where $e \in E(H)$ and the sets $W(v)$ are as usual) we have

$$|f| = \sum_{v \in e} |W(v)| = \sum_{v \in e} w(v) = N \sum_{v \in e} w^*(v) \geq N.$$

(In particular, every edge of H^w is non-empty.) Consequently, $|f| \geq \varepsilon |V(H^w)|$ where $\varepsilon = (\tau^*(H))^{-1}$. Certainly $\tau^*(H) \geq 1$, since $\delta(H) > 0$ and so $E(H) \neq \emptyset$; and hence $0 < \varepsilon \leq 1$. If $|E(H^w)| \leq 1$, then $\tau(H^w) \leq 1$ and so $\tau(H) \leq 1$, and since $\tau^*(H) \geq 1$ and $\delta(H) \geq 1$ the result is true. We may assume then that $|E(H^w)| \geq 2$. By (5.1) applied to H^w,

$$\tau(H^w) \leq 2\delta(H^w)\varepsilon^{-1} \log(11\varepsilon^{-1}).$$

But $\varepsilon^{-1} = \tau^*(H)$, and $\delta(H^w) \leq \delta(H)$ by (5.3), and $\tau(H^w) \geq \tau(H)$ as is easily seen; and so

$$\tau(H) \leq 2\delta(H)\tau^*(H) \log(11\tau^*(H))$$

as required. ∎

(5.5) *Let H be a hypergraph, and let $d \geq 1$ be an integer. Then $\delta < \binom{\lambda_d + 1}{d}$.*

[$\lambda_d(H)$ was defined in Section 1.]

Proof. Suppose that $\delta(H) \geq \binom{\lambda_d(H)+1}{d}$. Let B be a set of cardinality $\lambda_d(H)+1$, and let A be the set of all subsets of B of cardinality d. Then $\delta(H) \geq |\mathsf{A}|$, and we may therefore choose distinct u_A $(A \in \mathsf{A})$ in $V(H)$ such that for every $X \subseteq S$ there is an edge $e \in E(H)$ with $S \cap e = X$, where $S = \{u_A \ : \ A \in \mathsf{A}\}$. For each $b \in B$, let $X_b = \{u_A \ : \ A \in \mathsf{A},\ b \in A\}$. Then for each $b \in B$ there exists $e_b \in E(H)$ such that $S \cap e_b = X_b$. It follows that for all $A \in \mathsf{A}$ and $b \in B$, $u_A \in e_b$ if and only if $b \in A$. Now since the sets X_b $(b \in B)$ are all distinct, so are the edges e_b $(b \in B)$. Let $A \in \mathsf{A}$; then $\{e_b \ : \ b \in B,\ u_A \in e_b\} = \{e_b \ : \ b \in A\}$. Consequently, $\lambda_d(H) \geq |B|$, a contradiction. ∎

From (5.4) and (5.5) we deduce the following.

(5.6) *Let H be a hypergraph with every edge non-empty, and let $d \geq 1$ be an integer. Then*

$$\tau \leq 2\binom{\lambda_d + 1}{d} \tau^* \log(11\tau^*).$$

Let H be a hypergraph. We denote by H^t the hypergraph with $V(H^t) = E(H)$ and with

$$E(H^t) = \{\{e \in E(H) \ : \ e \in v\} \ : \ v \in V(H)\}.$$

Roughly, H^t is obtained from H by transposing the vertex/edge incidence matrix. However, we defined $E(H)$ to be a set rather than a multiset, and so all edges of a hypergraph are distinct. Thus, since there may be pairs of vertices of H in precisely the same edges, it is possible that $|E(H^t)| \neq |V(H)|$; and in general $(H^t)^t \neq H$.

From (5.6) and (4.2) we deduce our main result, the following.

(5.7) *Let H be a hypergraph with every edge non-empty, and let $d \geq 2$ be an integer. Then*

$$\tau \leq 11\lambda_d^d R(d, \lambda_d, \nu_{d-1})^d \log(8R(d, \lambda_d, \nu_{d-1})).$$

Proof. By (4.2) applied to H^t we deduce

$$\alpha^*(H^t) \leq \frac{(d+1)^{d+1}}{d! d^d} R(d, \phi_d(H^t), \alpha_{d-1}(H^t))^d.$$

But $\alpha^*(H^t) = \tau^*(H)$ by linear programming duality, $\phi_d(H^t) = \lambda_d(H)$, and $\alpha_{d-1}(H^t) = \nu_{d-1}(H)$. We deduce that

$$\tau^*(H) \leq \frac{(d+1)^{d+1}}{d! d^d} R(d, \lambda_d(H), \nu_{d-1}(H))^d.$$

The result follows from (5.6), after some arithmetic. ∎

By setting $d=2$ in (5.7) and using the fact that $R(2,r,s) \leq \binom{r+s}{r}$ (and hence $\log R(2,r,s) \leq r+s$) we obtain (1.1). In particular, it follows that if either ν or λ_2 is bounded above by a constant then τ is bounded above by a polynomial in λ_2 or ν respectively.

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